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DEFORMATION CHARACTERISTICS OF LAMINAR COMPOSITES UNDER NONLINEAR STRAINS

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It is shown that the type of governing relations in a composite can change, namely, a laminar composite formed by layers of physically linear materials under nonlinear strains should be described by nonlinear Hooke's law. Local stresses can be not proportional to elastic constants of the layers under nonlinear strains.

Key words: *laminar composites, macroscopic properties, nonlinear strains.*

Formulation of the Problem. Composites can possess macroscopic properties different from the properties of their constituents [1]. Examples of qualitative changes in composite properties, however, are few [2–4]. We consider a specimen of a laminar material of unit thickness (Fig. 1a) and the corresponding specimen of a homogeneous material (Fig. 1b). The requirement of infinite extension of the specimens along the x_1 and x_2 axes eliminates the problem of the edge effect. Let these specimens have identical strains "as a whole". We calculate their responses to these strains by determining the forces at the specimen boundaries. Our objective is to introduce the governing equations for the homogeneous specimen so that both specimens have identical responses to identical strains. We consider specimens infinite in the plane Ox_1x_2 (to avoid problems associated with edge effects) and extended from 0 to 1 along the Ox_3 axis.

Obtaining Averaged Governing Equations. For the homogeneous specimen, we consider the displacements

$$u_1 = v_{11}x_1 + v_{12}x_2 + v_{13}x_3,$$
$$u_2 = v_{12}x_1 + v_{22}x_2 + v_{23}x_3, \qquad u_3 = v_{13}x_1 + v_{23}x_2 + v_{33}x_3, \tag{1}$$

which satisfy the equilibrium equations for the homogeneous body and, for various v_{ij}, correspond to all types of basis deformations (extension along the axes and shear). We consider deformation of the laminar specimen with displacements (1) identical to those in the homogeneous specimen set on its boundary $x_3 = 0$ and $x_3 = 1$. We seek the solution of the problem of the elasticity theory in the form of a sum of homogeneous displacements (1) (deformations of the homogeneous specimen) and local displacements $v_i(x_3)$, which do not alter strains "as a whole":

$$U_1 = v_{11}x_1 + v_{12}x_2 + v_{13}x_3 + v_1(x_3),$$
$$U_2 = v_{12}x_1 + v_{22}x_2 + v_{23}x_3 + v_2(x_3), \qquad U_3 = v_{13}x_1 + v_{23}x_2 + v_{33}x_3 + v_3(x_3). \tag{2}$$

For displacements (1) and (2) at the specimen boundaries to be identical, the functions v_1, v_2, and v_3 should vanish at $x_3 = 0$ and $x_3 = 1$:

$$v_i(0) = v_i(1) = 0, \qquad i = 1, 2, 3. \tag{3}$$

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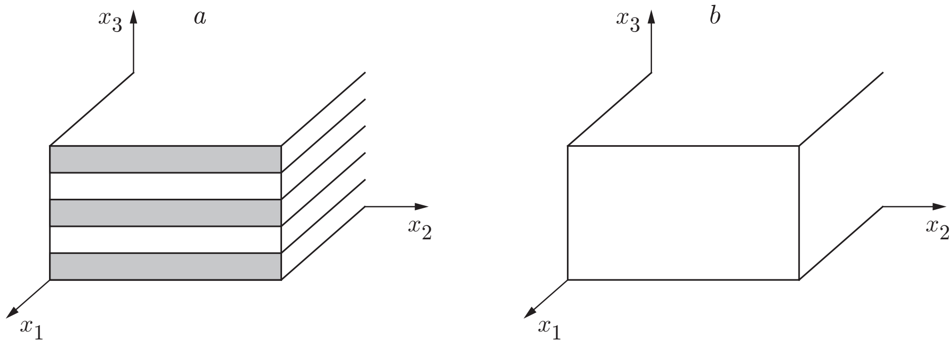

Fig. 1

We require satisfaction of equilibrium equations of the elasticity theory (we use nonlinear equations of the elasticity theory written in a nondeformed coordinate system):

$$(\sigma_{nk}(\delta_{\alpha k} + U_{\alpha,k}))_{,n} = 0, \qquad n, \alpha, k = 1, 2, 3. \tag{4}$$

We verify that functions of the form (2) can satisfy the equilibrium equations (4). We consider the nonlinear strains

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})/2. \tag{5}$$

The strains of the composite calculated by formulas (2) and (5) have the form

$$\begin{gathered}
\varepsilon_{11} = v_{11} + (v_{11}^2 + v_{12}^2 + v_{13}^2)/2, \qquad \varepsilon_{12} = v_{12} + (v_{11}v_{12} + v_{12}v_{22} + v_{13}v_{23})/2, \\
\varepsilon_{13} = v_1'/2 + v_{13} + (v_{11}v_{13} + v_{12}v_{23} + v_{13}v_{33} + v_1'v_{11} + v_2'v_{12} + v_3'v_{13})/2, \\
\varepsilon_{12} = \varepsilon_{21}, \qquad \varepsilon_{22} = v_{22} + (v_{12}^2 + v_{22}^2 + v_{23}^2)/2, \qquad \varepsilon_{13} = \varepsilon_{31}, \qquad \varepsilon_{23} = \varepsilon_{32}, \\
\varepsilon_{23} = v_2'/2 + v_{23} + (v_{12}v_{13} + v_{22}v_{23} + v_{23}v_{33} + v_1'v_{12} + v_2'v_{22} + v_3'v_{23})/2, \\
\varepsilon_{33} = v_3' + v_{33} + (v_{13}^2 + v_{23}^2 + v_{33}^2 + v_1'^2 + v_2'^2 + v_3'^2)/2 + v_1'v_{13} + v_2'v_{23} + v_3'v_{33}
\end{gathered} \tag{6}$$

(the prime denotes derivatives with respect to x_3).

We denote the strains of the homogeneous body calculated by formulas (1) and (5) as ε_{ij}^*:

$$\begin{gathered}
\varepsilon_{11}^* = \varepsilon_{11}, \qquad \varepsilon_{12}^* = \varepsilon_{12}, \qquad \varepsilon_{13}^* = v_{13} + (v_{11}v_{13} + v_{12}v_{23} + v_{13}v_{33})/2, \\
\varepsilon_{21}^* = \varepsilon_{21}, \qquad \varepsilon_{22}^* = \varepsilon_{22}, \qquad \varepsilon_{23}^* = v_{23} + (v_{12}v_{13} + v_{22}v_{23} + v_{23}v_{33})/2, \\
\varepsilon_{31}^* = \varepsilon_{31}, \qquad \varepsilon_{32}^* = \varepsilon_{32}, \qquad \varepsilon_{33}^* = v_{33} + (v_{13}^2 + v_{23}^2 + v_{33}^2)/2.
\end{gathered} \tag{7}$$

The values of ε_{ij}^* determined by formulas (7) are nonlinear strains "as a whole." They should be identical for the laminar and homogeneous bodies.

The following procedure is useful for solving the problem. We consider the products $\varepsilon_{ki}^* v_i'$:

$$\varepsilon_{ki}^* v_i' = v_{ki}v_i' + v_{kj}v_{ji}v_i' \quad \text{(summation over } j\text{)}. \tag{8}$$

Since v_j' are of the same order as ε_{ij}^* and v_{ij}, we can omit terms higher than those of the second order in Eqs. (8) as being small; then, the following equalities are valid:

$$\varepsilon_{ki}^* v_i' = v_{ki}v_i'. \tag{9}$$

Using equalities (9), we can write Eq. (6) in the form

$$\varepsilon_{11} = \varepsilon^*_{11}, \qquad \varepsilon_{12} = \varepsilon^*_{12}, \qquad \varepsilon_{13} = \varepsilon^*_{13} + v'_1/2 + (\varepsilon^*_{11}v'_1 + \varepsilon^*_{12}v'_2 + \varepsilon^*_{13}v'_3)/2,$$
$$\varepsilon_{22} = \varepsilon^*_{22}, \qquad \varepsilon_{21} = \varepsilon^*_{12}, \qquad \varepsilon_{23} = \varepsilon^*_{23} + v'_2/2 + (\varepsilon^*_{12}v'_1 + \varepsilon^*_{22}v'_2 + \varepsilon^*_{23}v'_3)/2, \tag{10}$$
$$\varepsilon_{33} = \varepsilon^*_{33} + v'_3 + \varepsilon^*_{13}v'_1 + \varepsilon^*_{23}v'_2 + \varepsilon^*_{33}v'_3 + (v'^2_1 + v'^2_2 + v'^2_3)/2.$$

Hence, we can express the nonlinear local strains ε_{ik} in the laminar composite via the nonlinear strains ε^*_{ik} "as a whole" and the derivatives of the local displacements v'_i.

We consider layers of physically linear materials. For these materials, Hooke's law in the nondeformed coordinate system is

$$\sigma_{11} = (\lambda + 2\mu)\varepsilon_{11} + \lambda(\varepsilon_{22} + \varepsilon_{33}) = (\lambda + 2\mu)\varepsilon^*_{11} + \lambda\varepsilon^*_{22}$$
$$+ \lambda(\varepsilon^*_{33} + v'_3 + \varepsilon^*_{13}v'_1 + \varepsilon^*_{23}v'_2 + \varepsilon^*_{33}v'_3) + \lambda(v'^2_1 + v'^2_2 + v'^2_3)/2,$$
$$\sigma_{22} = (\lambda + 2\mu)\varepsilon_{22} + \lambda(\varepsilon_{11} + \varepsilon_{33}) = (\lambda + 2\mu)\varepsilon^*_{22} + \lambda\varepsilon^*_{11} \tag{11}$$
$$+ \lambda(\varepsilon^*_{33} + v'_3 + \varepsilon^*_{13}v'_1 + \varepsilon^*_{23}v'_2 + \varepsilon^*_{33}v'_3) + \lambda(v'^2_1 + v'^2_2 + v'^2_3)/2,$$
$$\sigma_{12} = 2\mu\varepsilon_{12}, \qquad \sigma_{13} = 2\mu\varepsilon_{13}, \qquad \sigma_{21} = \sigma_{12}, \qquad \sigma_{23} = 2\mu\varepsilon_{23},$$
$$\sigma_{31} = \sigma_{13}, \qquad \sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22}) + (\lambda + 2\mu)\varepsilon_{33},$$

where σ_{ij} are the stresses in the nondeformed coordinate system.

Let us consider the equilibrium equations. Since all functions depend only on one variable x_3, the equilibrium equations (4) acquire the form

$$(\sigma_{nk}(\delta_{\alpha k} + U'_\alpha))' = 0. \tag{12}$$

Equations (12) yield the equalities

$$\sigma_{13}(1 + \varepsilon^*_{11}) + \sigma_{23}\varepsilon^*_{12} + \sigma_{33}(\varepsilon^*_{13} + v'_1) = C_{13} = \text{const},$$
$$\sigma_{13}\varepsilon^*_{12} + \sigma_{23}(1 + \varepsilon^*_{22}) + \sigma_{33}(\varepsilon^*_{23} + v'_2) = C_{23} = \text{const}, \tag{13}$$
$$\sigma_{13}\varepsilon^*_{13} + \sigma_{23}\varepsilon^*_{23} + \sigma_{33}(1 + \varepsilon^*_{33} + v'_3) = C_{33} = \text{const}.$$

The quantities C_{i3} have the meaning of stresses with the subscripts $i3$ and $3i$ in the deformed coordinate system ("true stresses").

Substituting strains (10) into the system of Hooke's law (11), we obtain

$$\sigma_{13} = 2\mu\varepsilon^*_{13} + \mu v'_1 + \mu(\varepsilon^*_{11}v'_1 + \varepsilon^*_{12}v'_2 + \varepsilon^*_{13}v'_3),$$
$$\sigma_{23} = 2\mu\varepsilon^*_{23} + \mu v'_2 + \mu(\varepsilon^*_{12}v'_1 + \varepsilon^*_{22}v'_2 + \varepsilon^*_{23}v'_3), \tag{14}$$
$$\sigma_{33} = \lambda(\varepsilon^*_{11} + \varepsilon^*_{22}) + (\lambda + 2\mu)[\varepsilon^*_{33} + v'_3 + \varepsilon^*_{13}v'_1 + \varepsilon^*_{23}v'_2 + \varepsilon^*_{33}v'_3 + (v'^2_1 + v'^2_2 + v'^2_3)/2].$$

We introduce the notation

$$A = \frac{\mu}{\lambda + 2\mu}, \qquad B = \frac{\lambda}{\lambda + 2\mu}, \qquad q_1 = \frac{C_{13}}{\mu}, \qquad q_2 = \frac{C_{23}}{\mu},$$
$$p_1 = \frac{C_{13}}{\lambda + 2\mu}, \qquad p_2 = \frac{C_{23}}{\lambda + 2\mu}, \qquad p_3 = \frac{C_{33}}{\lambda + 2\mu}. \tag{15}$$

Note, the quantities introduced in (15) are functions of the variable x_3.

Substituting (14) into (13), with allowance for notation (15), we obtain the following relations:

$$2A\varepsilon^*_{13} + Av'_1 + \varepsilon^*_{11}v'_1 + 2A\varepsilon^*_{12}v'_2 + (A + 1)\varepsilon^*_{13}v'_3 + \varepsilon^*_{13}\varepsilon^*_{33} + \varepsilon^*_{33}v'_1 + v'_1v'_3$$
$$+ \varepsilon^*_{11}\varepsilon^*_{13} + B\varepsilon^*_{13}\varepsilon^*_{22} + B\varepsilon^*_{22}v'_1 + 2A\varepsilon^*_{12}\varepsilon^*_{23} = p_1,$$

$$2A\varepsilon^*_{23} + Av'_2 + \varepsilon^*_{22}v'_2 + 2A\varepsilon^*_{12}v'_1 + (A+1)\varepsilon^*_{23}v'_3 + \varepsilon^*_{23}\varepsilon^*_{33} + \varepsilon^*_{33}v'_2 + v'_2v'_3 \tag{16}$$

$$+ \varepsilon^*_{22}\varepsilon^*_{23} + B\varepsilon^*_{23}\varepsilon^*_{11} + B\varepsilon^*_{11}v'_2 + 2A\varepsilon^*_{12}\varepsilon^*_{13} = p_2,$$

$$B\varepsilon^*_{11} + B\varepsilon^*_{22} + \varepsilon^*_{33} + v'_3 + (A+1)\varepsilon^*_{13}v'_1 + (A+1)\varepsilon^*_{23}v'_2 + 3\varepsilon^*_{33}v'_3 + (v'^2_1 + v'^2_2 + v'^2_3)/2$$

$$+ 2A\varepsilon^{*2}_{23} + 2A\varepsilon^{*2}_{13} + B\varepsilon^*_{11}\varepsilon^*_{33} + B\varepsilon^*_{22}\varepsilon^*_{33} + B\varepsilon^*_{11}v'_3 + B\varepsilon^*_{22}v'_3 + \varepsilon^{*2}_{33} + v'^2_3 = p_3.$$

System (16) relates the derivatives of the local displacements v'_i with strains "as a whole" ε^*_{ij}. We resolve Eq. (16) with respect to v'_i [with respect to these variables, Eq. (16) is a system of three nonlinear algebraic equations]. From the first two equations, we express v'_1 and v'_2 as functions of v'_3:

$$v'_1 = p_1/A - 2\varepsilon^*_{13} + \varepsilon^*_{13}(\varepsilon^*_{11} + \varepsilon^*_{33})/A - 2\varepsilon^*_{12}\varepsilon^*_{23} + B\varepsilon^*_{13}\varepsilon^*_{22}/A$$

$$- 2\varepsilon^*_{12}v'_2 + (1/A - 1)\varepsilon^*_{13}v'_3 - p_1(\varepsilon^*_{11} + \varepsilon^*_{33} + B\varepsilon^*_{22} + v'_3)/A^2,$$

$$v'_2 = p_2/A - 2\varepsilon^*_{23} + \varepsilon^*_{23}(\varepsilon^*_{22} + \varepsilon^*_{33})/A - 2\varepsilon^*_{12}\varepsilon^*_{13} + B\varepsilon^*_{11}\varepsilon^*_{23}/A \tag{17}$$

$$- 2\varepsilon^*_{12}v'_1 + (1/A - 1)\varepsilon^*_{23}v'_3 - p_2(B\varepsilon^*_{11} + \varepsilon^*_{33} + \varepsilon^*_{22} + v'_3)/A^2.$$

Since v'_1 and v'_2 are included into the third equation of (16) in combinations with ε^*_{ij}, we can eliminate terms of the second order of smallness in (17), because, together with ε^*_{ij}, they produce quantities of the third order. As a result, we obtain

$$v'_1 = p_1/A - 2\varepsilon^*_{13}, \qquad v'_2 = p_2/A - 2\varepsilon^*_{23}. \tag{18}$$

Substituting these values of v'_1 and v'_2 into the third equation of (16), we obtain the following equation for v'_3:

$$3v'^2_3/2 + v'_3(1 + 3\varepsilon^*_{33} + B\varepsilon^*_{11} + B\varepsilon^*_{22}) + \varepsilon^*_{33} + B(\varepsilon^*_{11} + \varepsilon^*_{22}) + B\varepsilon^*_{11}\varepsilon^*_{33} + B\varepsilon^*_{22}\varepsilon^*_{33} + \varepsilon^{*2}_{33}$$

$$- \varepsilon^*_{13}q_1 - \varepsilon^*_{23}q_2 + p_1\varepsilon^*_{13} + p_2\varepsilon^*_{23} + (q_1^2 + q_2^2)/2 - p_3 - \varepsilon^*_{13}q_1 - \varepsilon^*_{23}q_2 = 0. \tag{19}$$

Equation (19) is a quadratic equation with respect to v'_3. Its solution has the form

$$v'_3 = -(1/3 + \varepsilon^*_{33} + B\varepsilon^*_{11}/3 + B\varepsilon^*_{22}/3) + [1 + 3\varepsilon^{*2}_{33} - 4B(\varepsilon^*_{11} + \varepsilon^*_{22}) + B^2(\varepsilon^{*2}_{11} + 2\varepsilon^*_{11}\varepsilon^*_{22} + \varepsilon^{*2}_{22})$$

$$- 6p_1\varepsilon^*_{13} - 6p_2\varepsilon^*_{23} - 3(q_1^2 + q_2^2) + 6p_3 + 6\varepsilon^*_{13}q_1 + 6\varepsilon^*_{23}q_2]^{1/2}/3. \tag{20}$$

The second root is rejected because it is inconsistent with low strains.

Expression (20) is inconvenient for subsequent calculations (because of the radical in this expression). Neglecting terms of the second order, we expand the radical into the Taylor series in the neighborhood of the point $\varepsilon^*_{ij} = 0$, $p_i = 0$, $q_i = 0$. As a result, we obtain that the radical in (20), with accuracy to second-order small terms, equals

$$1 + 3p_3 - 2\,\frac{\lambda}{\lambda + 2\mu}\,(\varepsilon^*_{11} + \varepsilon^*_{22}) - \frac{3}{2}\,q_1^2 - \frac{3}{2}\,q_2^2 - \frac{9}{2}\,p_3^2 + \frac{3}{2}\,\varepsilon^{*2}_{33} - \frac{3}{2}\,\frac{\lambda^2}{(\lambda + 2\mu)^2}\,(\varepsilon^{*2}_{11} + \varepsilon^{*2}_{22})$$

$$- 3\varepsilon^*_{13}p_1 - 3\varepsilon^*_{23}p_2 + 6\,\frac{\lambda}{\lambda + 2\mu}\,(\varepsilon^*_{11}p_3 + \varepsilon^*_{22}p_3) - 3\frac{\lambda^2}{(\lambda + 2\mu)^2}\,\varepsilon^*_{11}\varepsilon^*_{22} + 3\varepsilon^*_{13}q_1 + 3\varepsilon^*_{23}q_2.$$

After that, Eq. (20) acquires the form

$$v'_3 = -\Big(\frac{1}{3} + \varepsilon^*_{33} + \frac{B}{3}\,\varepsilon^*_{11} + \frac{B}{3}\,\varepsilon^*_{22}\Big) + \frac{1}{3}\Big[1 + 3p_3 - 2\,\frac{\lambda}{\lambda + 2\mu}\,(\varepsilon^*_{11} + \varepsilon^*_{22})$$

$$- \frac{3}{2}\,q_1^2 - \frac{3}{2}\,q_2^2 - \frac{9}{2}\,p_3^2 + \frac{3}{2}\varepsilon^{*2}_{33} - \frac{3}{2}\,\frac{\lambda^2}{(\lambda + 2\mu)^2}\,(\varepsilon^{*2}_{11} + \varepsilon^{*2}_{22}) - 3\varepsilon^*_{13}p_1 - 3\varepsilon^*_{23}p_2$$

$$+ 6\,\frac{\lambda}{\lambda + 2\mu}\,(\varepsilon^*_{11}p_3 + \varepsilon^*_{22}p_3) - 3\,\frac{\lambda^2}{(\lambda + 2\mu)^2}\,\varepsilon^*_{11}\varepsilon^*_{22} + 3\varepsilon^*_{13}q_1 + 3\varepsilon^*_{23}q_2\Big]. \tag{21}$$

Formulas (18) and (21) yield an explicit expression for the derivatives of the local displacements v_i' in terms of the strains "as a whole" ε_{ij}^*. By virtue of the boundary conditions (3), we have $v_3(0) = v_3(1) = 0$. Therefore,

$$\int_0^1 v_3'(x_3)\,dx_3 = 0.$$

We introduce the mean value over the package thickness by the formula

$$\langle f\rangle = \int_0^1 f(x_3)\,dx_3.$$

Then, we can write $\langle v_3'\rangle = 0$.

We substitute v_3' into the last equality according to (21) and replace p_i and q_i by their expressions in terms of C_{13}, C_{23}, and C_{33} according to (15). By averaging the result, we obtain

$$C_{33}\Big\langle\frac{1}{\lambda+2\mu}\Big\rangle = \varepsilon_{33}^* + \Big\langle\frac{\lambda}{\lambda+2\mu}\Big\rangle(\varepsilon_{11}^*+\varepsilon_{22}^*) + \frac{1}{2}\,C_{13}^2\Big\langle\frac{1}{\mu^2}\Big\rangle + \frac{1}{2}\,C_{23}^2\Big\langle\frac{1}{\mu^2}\Big\rangle$$

$$+\frac{3}{2}\,C_{33}^2\Big\langle\frac{1}{(\lambda+2\mu)^2}\Big\rangle - \frac{1}{2}\,\varepsilon_{33}^{*2} + \frac{1}{2}\Big\langle\frac{\lambda^2}{(\lambda+2\mu)^2}\Big\rangle(\varepsilon_{11}^{*2}+\varepsilon_{22}^{*2}) - C_{13}\varepsilon_{13}^*\Big\langle\frac{1}{\mu}\Big\rangle - C_{23}\varepsilon_{23}^*\Big\langle\frac{1}{\mu}\Big\rangle$$

$$+ C_{13}\varepsilon_{13}^*\Big\langle\frac{1}{\lambda+2\mu}\Big\rangle + C_{23}\varepsilon_{23}^*\Big\langle\frac{1}{\lambda+2\mu}\Big\rangle - 2C_{33}\Big\langle\frac{\lambda}{(\lambda+2\mu)^2}\Big\rangle(\varepsilon_{11}^*+\varepsilon_{22}^*) + \Big\langle\frac{\lambda^2}{(\lambda+2\mu)^2}\Big\rangle\varepsilon_{11}^*\varepsilon_{22}^*. \qquad (22)$$

From the equality $\langle v_1'\rangle = 0$, in a similar manner, we obtain

$$C_{13}\Big\langle\frac{1}{\mu}\Big\rangle = 2\varepsilon_{13}^* + 2\varepsilon_{11}^*C_{13}\Big\langle\frac{1}{\mu}\Big\rangle - 2\varepsilon_{11}^*\varepsilon_{13}^* - \Big\langle\frac{\lambda}{\lambda+2\mu}\Big\rangle\varepsilon_{13}^*(\varepsilon_{22}^*+\varepsilon_{11}^*) + 2\varepsilon_{12}^*C_{23}\Big\langle\frac{1}{\mu}\Big\rangle$$

$$+\varepsilon_{13}^*C_{33}\Big\langle\frac{1}{\lambda+2\mu}\Big\rangle - 2\varepsilon_{12}^*\varepsilon_{23}^* - \varepsilon_{13}^*\varepsilon_{33}^* - \varepsilon_{13}^*C_{33}\Big\langle\frac{1}{\mu}\Big\rangle + C_{13}C_{33}\Big\langle\frac{1}{\mu^2}\Big\rangle. \qquad (23)$$

From the equality $\langle v_2'\rangle = 0$, we obtain

$$C_{23}\Big\langle\frac{1}{\mu}\Big\rangle = 2\varepsilon_{23}^* + 2\varepsilon_{12}^*C_{13}\Big\langle\frac{1}{\mu}\Big\rangle - 2\varepsilon_{12}^*\varepsilon_{13}^* - \Big\langle\frac{\lambda}{\lambda+2\mu}\Big\rangle\varepsilon_{23}^*(\varepsilon_{22}^*+\varepsilon_{11}^*)$$

$$+\varepsilon_{23}^*C_{33}\Big\langle\frac{1}{\lambda+2\mu}\Big\rangle + 2\varepsilon_{22}^*C_{23}\Big\langle\frac{1}{\mu}\Big\rangle - 2\varepsilon_{22}^*\varepsilon_{23}^* - \varepsilon_{23}^*\varepsilon_{33}^* - \varepsilon_{23}^*C_{33}\Big\langle\frac{1}{\mu}\Big\rangle + C_{23}C_{33}\Big\langle\frac{1}{\mu^2}\Big\rangle.$$

We represent the local stresses C_{ni} in the deformed coordinate system in the form [5] $C_{ni}^{\rm loc} = \sigma_{nk}(\delta_{ik}+U_{i,k})$. In particular, the following equalities are valid:

$$C_{11}^{\rm loc} = \sigma_{11} + \sigma_{11}U_{1,1} + \sigma_{12}U_{1,2} + \sigma_{13}U_{1,3},$$

$$C_{12}^{\rm loc} = \sigma_{12} + \sigma_{11}U_{2,1} + \sigma_{12}U_{2,2} + \sigma_{13}U_{2,3}, \quad C_{22}^{\rm loc} = \sigma_{22} + \sigma_{21}U_{2,1} + \sigma_{22}U_{2,2} + \sigma_{23}U_{2,3}. \qquad (24)$$

We average the local values $C_{ij}^{\rm loc}$ (24). With allowance for (2), we obtain

$$C_{11} = \langle C_{11}^{\rm loc}\rangle = \bar\sigma_{11}(1+\varepsilon_{11}^*) + \bar\sigma_{12}\varepsilon_{12}^* + \bar\sigma_{13}\varepsilon_{13}^*,$$

$$C_{12} = \langle C_{12}^{\rm loc}\rangle = \bar\sigma_{12}(1+\varepsilon_{22}^*) + \bar\sigma_{11}\varepsilon_{12}^* + \bar\sigma_{13}\varepsilon_{23}^*, \quad C_{22} = \langle C_{22}^{\rm loc}\rangle = \bar\sigma_{22}(1+\varepsilon_{22}^*) + \bar\sigma_{21}\varepsilon_{12}^* + \bar\sigma_{23}\varepsilon_{23}^*. \qquad (25)$$

We substitute σ_{ij} according to (11) and U_i according to (2) into (24). Then, we obtain the following expressions for the local stresses $C_{ij}^{\rm loc}$ $(i,j=1,2)$:

$$C_{11}^{\rm loc} = (\lambda+2\mu)\varepsilon_{11}^* + \lambda\varepsilon_{22}^* + (\lambda+2\mu)\varepsilon_{11}^{*2} + \lambda\varepsilon_{11}^*\varepsilon_{22}^* + \lambda\Big[\frac{C_{33}}{\lambda+2\mu} - \frac{\lambda}{\lambda+2\mu}\,(\varepsilon_{11}^*+\varepsilon_{22}^*)$$

$$-\frac{C_{33}^2}{(\lambda+2\mu)^2} + C_{33}\,\frac{\lambda}{(\lambda+2\mu)^2}\,(\varepsilon_{11}^*+\varepsilon_{22}^*) - C_{13}\,\frac{\varepsilon_{13}^*}{\lambda+2\mu} - C_{23}\,\frac{\varepsilon_{23}^*}{\lambda+2\mu}\Big]$$

$$+ C_{33}\,\frac{\lambda}{\lambda+2\mu}\,\varepsilon^*_{11} - \frac{\lambda^2}{\lambda+2\mu}\,(\varepsilon^*_{11}+\varepsilon^*_{22})\varepsilon^*_{11} + 2\mu\varepsilon^{*2}_{12} + \frac{C^2_{13}}{\mu} - C_{13}\varepsilon^*_{13},$$

$$C^{\rm loc}_{22} = (\lambda+2\mu)\varepsilon^*_{22} + \lambda\varepsilon^*_{11} + (\lambda+2\mu)\varepsilon^{*2}_{22} + \lambda\varepsilon^*_{11}\varepsilon^*_{22} + \lambda\Big[\frac{C_{33}}{\lambda+2\mu} - \frac{\lambda}{\lambda+2\mu}\,(\varepsilon^*_{11}+\varepsilon^*_{22}) \qquad (26)$$

$$-\frac{C^2_{33}}{(\lambda+2\mu)^2} + C_{33}\,\frac{\lambda}{(\lambda+2\mu)^2}\,(\varepsilon^*_{11}+\varepsilon^*_{22}) - C_{13}\,\frac{\varepsilon^*_{13}}{\lambda+2\mu} - C_{23}\,\frac{\varepsilon^*_{23}}{\lambda+2\mu}\Big]$$

$$+C_{33}\,\frac{\lambda}{\lambda+2\mu}\,\varepsilon^*_{22} - \frac{\lambda^2}{\lambda+2\mu}\,(\varepsilon^*_{11}+\varepsilon^*_{22})\varepsilon^*_{22} + 2\mu\varepsilon^{*2}_{12} + \frac{C^2_{23}}{\mu} - C_{23}\varepsilon^*_{23},$$

$$C^{\rm loc}_{12} = 2\mu\varepsilon^*_{12} + (\lambda+2\mu)(\varepsilon^*_{11}+\varepsilon^*_{22})\varepsilon^*_{12} + \varepsilon^*_{12}\lambda\Big[\frac{C_{33}}{\lambda+2\mu} - \frac{\lambda}{\lambda+2\mu}\,(\varepsilon^*_{11}+\varepsilon^*_{22})\Big] + \frac{C_{13}C_{23}}{\mu} - C_{13}\varepsilon^*_{23}.$$

We perform averaging in (26) and take into account that $\langle C^{\rm loc}_{ij}\rangle = C_{ij}$ according to (25). As a result, we obtain the following equations for $\bar\sigma_{11}$, $\bar\sigma_{22}$, and $\bar\sigma_{12}$:

$$\bar\sigma_{11}(1+\varepsilon^*_{11}) + \bar\sigma_{12}\varepsilon^*_{12} + \bar\sigma_{13}\varepsilon^*_{13} = \langle\lambda+2\mu\rangle\varepsilon^*_{11} + \langle\lambda\rangle\varepsilon^*_{22} + \langle\lambda+2\mu\rangle\varepsilon^{*2}_{11} + \langle\lambda\rangle\varepsilon^*_{11}\varepsilon^*_{22}$$

$$+\Big\langle\frac{\lambda}{\lambda+2\mu}\Big\rangle\bar\sigma_{33}(1+\varepsilon^*_{33}) - \Big\langle\frac{\lambda^2}{\lambda+2\mu}\Big\rangle(\varepsilon^*_{11}+\varepsilon^*_{22}) - \bar\sigma^2_{33}\Big\langle\frac{\lambda}{(\lambda+2\mu)^2}\Big\rangle + \Big\langle\frac{\lambda^2}{(\lambda+2\mu)^2}\Big\rangle\bar\sigma_{33}(\varepsilon^*_{11}+\varepsilon^*_{22})$$

$$+\Big\langle\frac{\lambda}{\lambda+2\mu}\Big\rangle\bar\sigma_{33}\varepsilon^*_{11} - (\varepsilon^*_{11}+\varepsilon^*_{22})\varepsilon^*_{11}\Big\langle\frac{\lambda^2}{\lambda+2\mu}\Big\rangle + 2\langle\mu\rangle\varepsilon^{*2}_{12} + \bar\sigma^2_{13}\Big\langle\frac{1}{\mu}\Big\rangle - \bar\sigma_{13}\varepsilon^*_{13},$$

$$\bar\sigma_{22}(1+\varepsilon^*_{22}) + \bar\sigma_{12}\varepsilon^*_{12} + \bar\sigma_{23}\varepsilon^*_{23} = \langle\lambda+2\mu\rangle\varepsilon^*_{22} + \langle\lambda\rangle\varepsilon^*_{11} + \langle\lambda+2\mu\rangle\varepsilon^{*2}_{22} + \langle\lambda\rangle\varepsilon^*_{11}\varepsilon^*_{22} \qquad (27)$$

$$+\Big\langle\frac{\lambda}{\lambda+2\mu}\Big\rangle\bar\sigma_{33}(1+\varepsilon^*_{33}) - \Big\langle\frac{\lambda^2}{\lambda+2\mu}\Big\rangle(\varepsilon^*_{11}+\varepsilon^*_{22}) - \bar\sigma^2_{33}\Big\langle\frac{\lambda}{(\lambda+2\mu)^2}\Big\rangle + \Big\langle\frac{\lambda^2}{(\lambda+2\mu)^2}\Big\rangle\bar\sigma_{33}(\varepsilon^*_{11}+\varepsilon^*_{22})$$

$$+\Big\langle\frac{\lambda}{\lambda+2\mu}\Big\rangle\bar\sigma_{33}\varepsilon^*_{22} - (\varepsilon^*_{11}+\varepsilon^*_{22})\varepsilon^*_{22}\Big\langle\frac{\lambda^2}{\lambda+2\mu}\Big\rangle + 2\langle\mu\rangle\varepsilon^{*2}_{12} + \sigma^2_{23}\Big\langle\frac{1}{\mu}\Big\rangle - \bar\sigma_{23}\varepsilon^*_{23},$$

$$\bar\sigma_{12}(1+\varepsilon^*_{22}) + \bar\sigma_{11}\varepsilon^*_{12} + \bar\sigma_{13}\varepsilon^*_{23} = 2\langle\mu\rangle\varepsilon^*_{12} + \langle\lambda+2\mu\rangle(\varepsilon^*_{11}+\varepsilon^*_{22})\varepsilon^*_{12}$$

$$+\Big\langle\frac{\lambda}{\lambda+2\mu}\Big\rangle\varepsilon^*_{12}\bar\sigma_{33} - (\varepsilon^*_{11}+\varepsilon^*_{22})\varepsilon^*_{12}\Big\langle\frac{\lambda^2}{\lambda+2\mu}\Big\rangle + \bar\sigma_{13}\bar\sigma_{23}\Big\langle\frac{1}{\mu}\Big\rangle - \bar\sigma_{13}\varepsilon^*_{23}.$$

Substituting the values of C_{13}, C_{23}, and C_{33} from (12) into (22) and (23), we obtain the following relations between the stresses $\bar\sigma_{ij}$ in the nondeformed coordinate system and the strains "as a whole" ε^*_{ij}:

$$0 = -\bar\sigma_{33}\Big\langle\frac{1}{\lambda+2\mu}\Big\rangle + \varepsilon^*_{33} + \Big\langle\frac{\lambda}{\lambda+2\mu}\Big\rangle(\varepsilon^*_{11}+\varepsilon^*_{22})$$

$$+\frac{1}{2}\,\bar\sigma_{13}\Big(\bar\sigma_{13}\Big\langle\frac{1}{\mu^2}\Big\rangle - 2\varepsilon^*_{13}\Big\langle\frac{1}{\mu}\Big\rangle\Big) + \frac{1}{2}\bar\sigma_{23}\Big(\bar\sigma_{23}\Big\langle\frac{1}{\mu^2}\Big\rangle - 2\varepsilon^*_{23}\Big\langle\frac{1}{\mu}\Big\rangle\Big)$$

$$+\frac{1}{2}\Big[\bar\sigma^2_{33}\Big\langle\frac{1}{(\lambda+2\mu)^2}\Big\rangle - 2\bar\sigma_{33}(\varepsilon^*_{11}+\varepsilon^*_{22})\Big\langle\frac{\lambda}{(\lambda+2\mu)^2}\Big\rangle + \Big\langle\frac{\lambda^2}{(\lambda+2\mu)^2}\Big\rangle(\varepsilon^*_{11}+\varepsilon^*_{22})^2 - \varepsilon^{*2}_{33}\Big]$$

$$+\bar\sigma_{33}\Big[\bar\sigma_{33}\Big\langle\frac{1}{(\lambda+2\mu)^2}\Big\rangle - \varepsilon^*_{11}\Big\langle\frac{\lambda}{(\lambda+2\mu)^2}\Big\rangle - \varepsilon^*_{22}\Big\langle\frac{\lambda}{(\lambda+2\mu)^2}\Big\rangle - \varepsilon^*_{33}\Big\langle\frac{1}{\lambda+2\mu}\Big\rangle\Big],$$

$$0 = \bar\sigma_{13}\Big\langle\frac{1}{\mu}\Big\rangle - 2\varepsilon^*_{13} + \varepsilon^*_{11}\Big(2\varepsilon^*_{13} - \bar\sigma_{13}\Big\langle\frac{1}{\mu}\Big\rangle\Big) - \bar\sigma_{33}\Big(\bar\sigma_{13}\Big\langle\frac{1}{\mu^2}\Big\rangle - 2\varepsilon^*_{13}\Big\langle\frac{1}{\mu}\Big\rangle\Big) \qquad (28)$$

$$-\varepsilon^*_{13}\Big[\bar\sigma_{33}\Big\langle\frac{1}{\lambda+2\mu}\Big\rangle - \varepsilon^*_{33} - \varepsilon^*_{11}\Big\langle\frac{\lambda}{\lambda+2\mu}\Big\rangle - \varepsilon^*_{22}\Big\langle\frac{\lambda}{\lambda+2\mu}\Big\rangle\Big] - \varepsilon^*_{12}\Big(\bar\sigma_{23}\Big\langle\frac{1}{\mu}\Big\rangle - 2\varepsilon^*_{23}\Big),$$

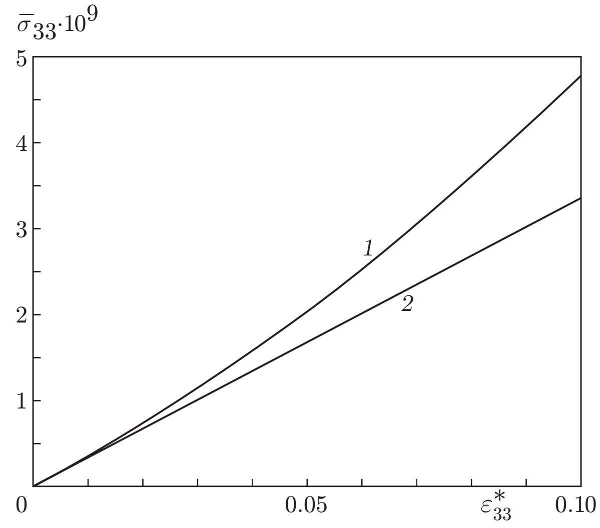

Fig. 2

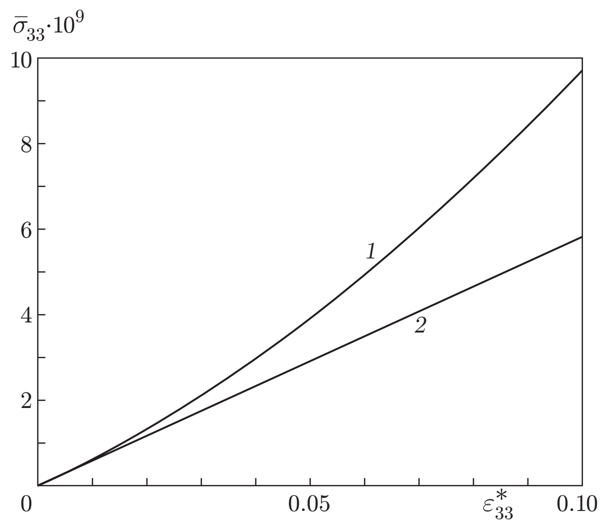

Fig. 3

$$0 = \bar{\sigma}_{23}\Big\langle \frac{1}{\mu}\Big\rangle - 2\varepsilon^*_{23} + \varepsilon^*_{22}\Big(2\varepsilon^*_{23} - \bar{\sigma}_{23}\Big\langle \frac{1}{\mu}\Big\rangle\Big) - \bar{\sigma}_{33}\Big(\bar{\sigma}_{23}\Big\langle \frac{1}{\mu^2}\Big\rangle - 2\varepsilon^*_{23}\Big\langle \frac{1}{\mu}\Big\rangle\Big)$$

$$- \varepsilon^*_{23}\Big[\bar{\sigma}_{33}\Big\langle \frac{1}{\lambda+2\mu}\Big\rangle - \varepsilon^*_{33} - \varepsilon^*_{11}\Big\langle \frac{1}{\lambda+2\mu}\Big\rangle - \varepsilon^*_{22}\Big\langle \frac{1}{\lambda+2\mu}\Big\rangle\Big] - \varepsilon^*_{12}\Big(\bar{\sigma}_{13}\Big\langle \frac{1}{\mu}\Big\rangle - 2\varepsilon^*_{13}\Big).$$

Equations (27) and (28) are the governing equations for the laminar specimen "as a whole," written in the nondeformed coordinate system in an implicit form. They relate the stresses $\bar{\sigma}_{ij}$ in the nondeformed coordinate system and the strains "as a whole" ε^*_{ij}.

Homogeneous Material. For $\lambda = \text{const}$ and $\mu = \text{const}$, the material in each layer is homogeneous, and the relation between the strain and stress should be linear. Indeed, the averaged law (27), (28) in this case becomes linear.

Linear Case. For low strains, eliminating second-order terms in Eqs. (27) and (28), we obtain the linear averaged Hooke's law. It coincides with the averaged Hooke's law for the linear-elastic laminar composite [4].

Investigation of Averaged Governing Relations. Equations (27) and (28) relate the averaged stresses $\bar{\sigma}_{ij}$ and strains "as a whole" ε^*_{ij}. The strains ε^*_{ij} are calculated by the nonlinear theory [relations (27) and (28) are Hooke's law for the laminar package "as a whole"). The local (for each material) Hooke's law (11) is physically linear. Law (27) and (28) in the general case is nonlinear, because it is possible to choose λ and μ so that (27), (28) are not perfect squares.

As an example, we consider the case of uniaxial strain: $\varepsilon^*_{33} \neq 0$, and the remaining $\varepsilon^*_{ij} = 0$. In this case, all shear stresses $\bar{\sigma}_{ij}$, except for $\bar{\sigma}_{33}$, equal zero, and only one equation remains from relations (27), (28):

$$-\bar{\sigma}_{33}\Big\langle \frac{1}{\lambda+2\mu}\Big\rangle + \varepsilon^*_{33} - \frac{1}{2}\Big(\varepsilon^{*2}_{33} - \bar{\sigma}^2_{33}\Big\langle \frac{1}{(\lambda+2\mu)^2}\Big\rangle\Big) + \bar{\sigma}_{33}\Big(\bar{\sigma}_{33}\Big\langle \frac{1}{(\lambda+2\mu)^2}\Big\rangle - \varepsilon^*_{33}\Big\langle \frac{1}{\lambda+2\mu}\Big\rangle\Big) = 0.$$

This is a quadratic equation with respect to $\bar{\sigma}_{33}$. Solving this equation, we obtain the stress $\bar{\sigma}_{33}$ as a function of the strain ε^*_{33}. The dependence of $\bar{\sigma}_{33}$ on ε^*_{33} is plotted in Figs. 2 and 3 for different composites of two species (curve 1). The data in Fig. 2 are shown for the following parameters of the layers: $E_1 = 10$, $E_2 = 110$ (10^9 Pa), $\nu_1 = 0.25$, $\nu_2 = 0.3$, and relative thicknesses of the layers $h_1 = 0.3$ and $h_2 = 0.7$. The data in Fig. 3 are given for $E_1 = 10$, $E_2 = 110$ (10^9 Pa), $\nu_1 = \nu_2 = 0.15$, and relative thicknesses of the layers $h_1 = 0.1$ and $h_2 = 0.9$. The straight line 2 shows the dependence of $\bar{\sigma}_{33}$ on ε^*_{33} for the linear-elastic composite with the same parameters of the layers (the dependence was calculated by formulas from [4]).

The calculations show that the dependence $\bar{\sigma}_{33}(\varepsilon^*_{33})$ almost coincides with the linear Hooke's law for $\varepsilon^* < 0.05$ and starts to deviate from the linear law at $\varepsilon^* > 0.05$; the deviation can reach 10% in the interval 0.05–0.1.

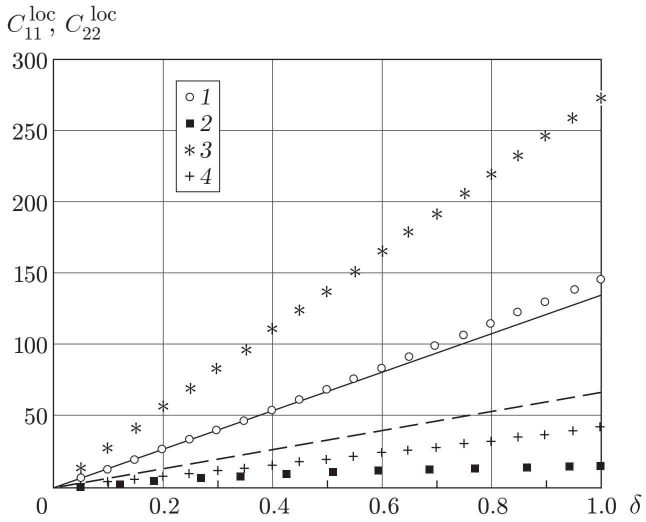

Fig. 4

The deviation from the nonlinear law increases with increasing ratio of Young's moduli of the layers and with decreasing thickness of the soft layer. The reason is the high strains in the soft layer.

The observed influence of both material characteristics of the layers and strains on the averaged characteristics of the composites is most obvious in the above-considered case of linear-elastic layers. The effect also holds for nonlinear-elastic layers. Indeed, the nonlinear stress-strain relations can be approximated by linear relations in the neighborhood of specified strains, and the effect is valid in this case. If nonzero strains are taken as the specified strains, the problem of the theory with initial stresses should be averaged [6].

Investigation of Local Stresses. Formulas (26) yield an expression of local stresses via strains "as a whole". We consider the case of biaxial strain in the plane of the layers. Let the stresses applied to the composite have the form $C_{11} \neq 0$, $C_{22} \neq 0$, and the remaining stresses be $C_{ij} = 0$. Averaging (26), in the case of biaxial strain, we determine the dependence of the mean stresses C_{11} and C_{22} on the strains ε_{11}^* and ε_{22}^* from the following system of algebraic equations:

$$C_{11} = \langle \lambda + 2\mu \rangle \varepsilon_{11}^* + \langle \lambda \rangle \varepsilon_{22}^* + \langle \lambda + 2\mu \rangle \varepsilon_{11}^{*2} - \Big\langle \frac{\lambda^2}{\lambda + 2\mu} \Big\rangle (\varepsilon_{11}^* + \varepsilon_{22}^*) + \langle \lambda \rangle \varepsilon_{11}^* \varepsilon_{22}^* - \Big\langle \frac{\lambda^2}{\lambda + 2\mu} \Big\rangle (\varepsilon_{11}^* + \varepsilon_{22}^*) \varepsilon_{11}^*,$$

$$C_{22} = \langle \lambda + 2\mu \rangle \varepsilon_{22}^* + \langle \lambda \rangle \varepsilon_{11}^* + \langle \lambda + 2\mu \rangle \varepsilon_{22}^{*2} - \Big\langle \frac{\lambda^2}{\lambda + 2\mu} \Big\rangle (\varepsilon_{11}^* + \varepsilon_{22}^*) + \langle \lambda \rangle \varepsilon_{11}^* \varepsilon_{22}^* - \Big\langle \frac{\lambda^2}{\lambda + 2\mu} \Big\rangle (\varepsilon_{11}^* + \varepsilon_{22}^*) \varepsilon_{22}^*. \quad (29)$$

To calculate the local stresses $C_{ij}^{\rm loc}$ by formulas (26), we have to know ε_{11}^* and ε_{22}^*, which are found by solving (29) with respect to these quantities. We seek the solution by expanding into a series in terms of the quantity δ, which has the order of strain. We retain terms of the series of the second or lower order in terms of δ. The solution has the form

$$\delta = \frac{C_{11}}{\langle \lambda + 2\mu - \lambda^2/(\lambda + 2\mu) \rangle},$$

$$\varepsilon_{11}^* = \delta\, \frac{1 - \gamma t}{1 - t^2} + \delta^2\, \frac{\gamma^2 t - \gamma t^2 + \gamma t - 1}{(1 - t^2)^2}, \qquad \varepsilon_{22}^* = \delta\, \frac{\gamma - t}{1 - t^2} + \delta^2\, \frac{-\gamma^2 - \gamma t^2 + t + \gamma t}{(1 - t^2)^2}, \quad (30)$$

where $\gamma = C_{22}/C_{11}$ and $t = \langle \lambda - \lambda^2/(\lambda + 2\mu) \rangle / \langle \lambda + 2\mu - \lambda^2/(\lambda + 2\mu) \rangle$.

For a composite formed by layers of two materials, we determine the local stresses in each layer as functions of the mean stresses C_{11} and C_{22}. Substituting (30) into (26), we obtain the following expressions for $C_{11}^{\rm loc}$ and $C_{22}^{\rm loc}$:

$$C_{11}^{\mathrm{loc},i} = \frac{\delta^2\gamma(1-\gamma)(1+t)}{(1-t^2)^2}\Big\{-\Big[\lambda_i + 2\mu_i - \frac{\lambda_i^2}{\lambda_i+2\mu_i}\Big]t + \lambda_i - \frac{\lambda_i^2}{\lambda_i+2\mu_i}\Big\}$$

$$+\frac{\delta}{1-t^2}\Big\{\Big[\lambda_i + 2\mu_i - \frac{\lambda_i^2}{\lambda_i+2\mu_i}\Big](1-\gamma t) + \Big[\lambda_i - \frac{\lambda_i^2}{\lambda_i+2\mu_i}\Big](\gamma - t)\Big\},$$

$$C_{22}^{\mathrm{loc},i} = \frac{\delta^2(1-\gamma)(1+t)}{(1-t^2)^2}\Big\{\Big[\lambda_i + 2\mu_i - \frac{\lambda_i^2}{\lambda_i+2\mu_i}\Big]t - \lambda_i - \frac{\lambda_i^2}{\lambda_i+2\mu_i}\Big\} \qquad (31)$$

$$+\frac{\delta}{1-t^2}\Big\{\Big[\lambda_i + 2\mu_i - \frac{\lambda_i^2}{\lambda_i+2\mu_i}\Big](\gamma - t) + \Big[\lambda_i - \frac{\lambda_i^2}{\lambda_i+2\mu_i}\Big](1-\gamma t)\Big\}$$

($i = 1,\ 2$ are the numbers of the layers).

For $C_{22} \neq 0$ and $C_{22} \neq C_{11}$, the local stresses are quadratic functions of the mean stress C_{11}.

The dependences of C_{11}^{loc} and C_{22}^{loc} on δ in each layer are plotted in Fig. 4 for $E_1 = 140$, $E_2 = 14$, $\nu_1 = 0.1$, $\nu_2 = 0.4$, and $h = 0.4$. Points 1 and 2 refer to C_{11}^{loc} in the first and second materials and points 3 and 4 refer to C_{22}^{loc} in the first and second materials, respectively; the solid and dashed curves are graphs of $C_{11} = \langle C_{11}^{\mathrm{loc}}\rangle$ and $C_{22} = \langle C_{22}^{\mathrm{loc}}\rangle$, respectively.

Note, for $\gamma = 0$, C_{11}^{loc} is not quadratic, whereas C_{22}^{loc} is quadratic.

In the linear-elastic composite, the stresses in the layers are proportional to rigidity of the layers and to mean stresses [1, 4], i.e., for the linear-elastic composite, formulas (31) contain only terms linear with respect to δ. Appearance of a quadratic term in the nonlinear case means that the distribution of stresses between the composite layers acquires a qualitatively different character if the strains become high.

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